Quiz 4 – Solutions

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1. Complete the partial sentence into a precise definition for, or a precise mathematical characterization of, the italicized term:

   Suppose $m, n \in \mathbb{Z}_{>0}$. A *linear transformation* $T : \mathbb{R}^m \to \mathbb{R}^n$ is ...

   Solution: A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a *linear transformation* if, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$,

   $$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{and} \quad T(\alpha \mathbf{u}) = \alpha\, T(\mathbf{u}).$$

   Equivalently, for all $a, b \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$,

   $$T(a\mathbf{u} + b\mathbf{v}) = a\, T(\mathbf{u}) + b\, T(\mathbf{v}).$$

2. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) The function $R : \mathbb{R}^2 \to \mathbb{R}^2$ that sends each vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to its reflection $\begin{bmatrix} y \\ x \end{bmatrix}$ over the line $y = x$ is a linear transformation.

   Solution: TRUE. For $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ define

   $$R(\mathbf{u}) = \begin{bmatrix} y \\ x \end{bmatrix}.$$

   Then for any $\mathbf{u}, \mathbf{v}$ and $\alpha \in \mathbb{R}$,

   $$R(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} u_2 + v_2 \\ u_1 + v_1 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = R(\mathbf{u}) + R(\mathbf{v}),$$

   and

   $$R(\alpha \mathbf{u}) = \begin{bmatrix} \alpha u_2 \\ \alpha u_1 \end{bmatrix} = \alpha \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \alpha\, R(\mathbf{u}).$$

   Thus R is linear. (Equivalently, $R(\mathbf{u}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}$.)

(b) The function $D : \mathbb{R}^2 \to \mathbb{R}$ that assigns to every point in the plane its distance from the origin is a linear transformation.

Solution: FALSE. Let $D(x, y) = \sqrt{x^2 + y^2}$. Take $\mathbf{v} = (1, 0)$ and $\alpha = -1$. Then

$$D(\alpha\mathbf{v}) = D(-1, 0) = 1 \neq -1 = -D(\mathbf{v}),$$

so homogeneity fails. (In fact $D(\alpha\mathbf{v}) = |\alpha|D(\mathbf{v})$.) Hence D is not linear.

(c) Let $\mathcal{D}(\mathbb{R})$ denote the set of differentiable functions on $\mathbb{R}$. If $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{D}(\mathbb{R})$, then

- $f + g \in \mathcal{D}(\mathbb{R})$, and
- $\alpha f \in \mathcal{D}(\mathbb{R})$.

Solution: TRUE. Fix $x \in \mathbb{R}$. Since f and g are differentiable at x,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \qquad g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

exist. Then

$$\frac{(f+g)(x+h) - (f+g)(x)}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \xrightarrow[h \to 0]{} f'(x) + g'(x),$$

so $f + g$ is differentiable with $(f + g)' = f' + g'$. Similarly,

$$\frac{(\alpha f)(x+h) - (\alpha f)(x)}{h} = \alpha \frac{f(x+h) - f(x)}{h} \xrightarrow[h \to 0]{} \alpha f'(x),$$

so αf is differentiable with $(\alpha f)' = \alpha f'$. As x was arbitrary, both results hold on $\mathbb{R}$.